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In search of some unifying symmetry through quantum groups in integrable and conformal theories

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Abstract : The basic features of integrable and conformal theories are reviewed Some common symmetries given through the underlying quantum group structures are shown to play an important role in both the theories. Exploiting the quantum group related Sklyanin-like algebras a systematic way for generating integrable models is presented. The relationship between integrable and conformal models is highlighted examining the metamorphosis of their transition.

Keywords : Integrable and conformal theories, their common symmetry, quantum group structures

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1. Introduction

Time and again in the history of scientific progress we had ample occassions to convince ourselves that the harmony in nature seems to be expressed through its preference for the unity among its diversities. Hence the search for some unifying theories continues in all scientific fields. In recent years we have once again witnessed the emergence of extremely interesting interconnections between seemingly diverse subjects like quantum and statistical systems; integrable systems, quantum groups and conformal theories; Yang-Baxter equation, braid group, fractional statistics etc. Among these variety of interesting topics we would like to focus in particular on the linkage between integrable and conformal theories. In spite of the recent serious studies the complete understanding of this relationship between these two most fascinating branches of two-dimensional theories is yet to be accomplished. Our main emphasis here, beside reviewing the basic features of these two theories, would be to explore some possible unifying scheme for generating integrable models, as well as to understand the metamorphosis of transition of integrable systems to conformal theories and also to focus on some common symmetries of these systems given through the underlying quantum groups.

Section 2 reviews the integrable systems (IS) exploring the role played by the quantum group structures. Section 3 gives a brief account of the conformal field theory (CFT) stressing on its quantum group symmetry, while section 4 discusses the interrelation between these two theories. Section 5 is the concluding section.

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2. Integrable systems and its unifying symmetry through quantum group structure

During the last two decades, nonlinear integrable systems have emerged as an exclusive class possessing a rich collection of representative common features [1]. The range of such systems covers already a vast number of models extending from discrete to continuum as well as from classical to quantum and statistical models [2]. Nonlinear Schrödinger equation (NLSE), derivative NLSE, spin models, Toda chain, sine-Gordon model (SG), Liouville model etc. are only a few well known examples of this class. Presently a vast number of classical field models come under the category of ***completely integrable*** Hamiltonian systems, represented by the exclusive properties like the existence of canonical action-angle variables and a set of infinite number of conserved quantities. These conserved quantities are expressed through action variables only and thus are always in involution. With respect to their symplectic structures such models may be classified into two broad subclasses, e.g. *ultralocal* [1] with standard δ-function canonical poisson brackets (PB) and *nonultralocal* [3] with PB's given through derivatives of δ-functions. Until now only the ultralocal models allow the quantum generalisation and exact solution through quantum inverse scattering transform (QIST) and hence we would concentrate here only on this particular class of models.

For the quantum integrability one requires the existence of a *R-matrix* solution satisfying the crucial quantum *Yang-Baxter equation* (QYBE) given by

$$R(\lambda,\mu)\, T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda)\, R(\lambda,\mu), \tag{2.1}$$

T being the scattering matrix operator with the elements

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ -B^{\dagger}(\lambda) & A^{\dagger}(\lambda) \end{pmatrix}$$

with R acting nontrivially on the space $V_{1/2} \otimes V_{1/2}$. Strictly speaking (2.1) is valid only for finite interval $[-L, L]$ and should be properly modified at $L \to \infty$. However, we would not be involved here into such details refering the readers to ref. [1]. Note that the QYBE, in fact, represents some specific 'commutation' relations among the operator elements of $T(\lambda)$ only written in the matrix form. An important consequence of (2.1) $[trT(\lambda), trT(\mu)] = 0$, easily found by taking tr from both the sides of the equation,ensures the integrability of the system. The other relation of the form

$$A(\lambda)B(\mu) = \tilde{a}(\mu,\lambda)B(\mu)A(\lambda) + f(\lambda)\delta(\lambda-\mu)B(\mu)A(\lambda)$$

derivable from the QYBE is equally important, since defining the vaccum state $|0>$ as $A(\lambda)|0> = |0>$ one may construct the N-particle state as $|N> = \prod_{i=1}^{N} B(\mu_i)|0>$. Such states turn out to be the eigenstates $A(\lambda)|N> = \tilde{a}(\lambda)|N>$ with $\tilde{a}(\lambda) = \prod_{i=1}^{N} a(\lambda,\mu_i)$ representing the eigenvalues of the operators of integrals of motion including the Hamiltonian. This method using the algebraic Bethe ansatz forms the basic principle of QIST. For solving the quantum field theories the model is usually defined on a lattice to avoid short distance singularities. As a consequence the monodromy matrix may be represented as $T_L(\lambda) = \lim_{\Delta \to 0} \Pi^1{}_{n=2N} L_n(\lambda)$, with $N = \frac{L}{\Delta}$, where Δ is the lattice constant and $L_n \to 1 + \Delta U^q(x, \lambda)$ is the lattice regularised version of the quantum *Lax operator* $U^q(x, \lambda)$ corresponding to the field-model. Using the ultralocal property $[L_n, L_m] = 0$ for $n \neq m$ one finds that eq. (2.1) is equivalent to its local version

$$R(\lambda, \mu) L_n(\lambda) \otimes L_n(\mu) = L_n(\mu) \otimes L_n(\lambda) R(\lambda, \mu). \tag{2.2}$$

The classical case is recovered at $h \to 0$, when the quantum commutator between fundamental fields are replaced by the canonical Poisson bracket (PB) reducing (2.2) to the relation

$$\{U(x, \lambda) \otimes U(y, \mu)\} = [r(\lambda, \mu), U(x, \lambda) \otimes 1 + 1 \otimes U(y, \mu)] \, \delta(x - y). \tag{2.3}$$

through the *classical r-matrix* solution. The classical analogue of (2.1) gives the PB relations between the elements of $T(\lambda)$. In particular $\{a(\lambda), a(\mu)\} = 0$ ensures $\ln a(\lambda)$ to be related to the *action* variables, while the relation $\{\ln a(\lambda), \ln b(\mu)\} = f(\lambda)\delta(\lambda - \mu)$ links $\ln b$ with the *angle* variables. Using the analytic properties of $a(\lambda)$, it may be expanded as

$$\ln a(\lambda) = \sum_{n=0}^{\infty} \frac{C_n}{\lambda^n}, \text{ for } \lambda \to \infty = \sum_{n=0}^{\infty} D_n \Lambda^n, \text{ for } \lambda \to 0, \tag{2.4a, b}$$

where C_n, D_n obviously represent infinite series of conserved quantities in involution. Thus the classical YBE and the existence of r-matrix are essential criteria for classical integrability of such systems.

However, inspite of an impressive achievement of this theory over the past twenty years, formulation of some underlying symmetry principle for constructing all such models in a systematic way is yet to be accomplished. Likewise why only certain models with specific nonlinearities happen to be integrable, seems also not well understood. Similarly QISF, though an enormously powerful method for solving quantum models, again does not answer properly why the quantum R-matrices related to spin chains as well as to a variety of field models should be identical. Such important unanswered questions should serve as enough motivations for investigating the above problems, as we have attempted here.

Recently, algebraic structures like quantum group and Sklyanin algebra have been studied extensively in connection with braid groups, conformal field theory, spin systems etc. [3]. We find surprisingly that they also play a significant role relevant to the above posed problems and at the same time define a crucial underlying common symmetry in integrable as well as conformal theories. Let us first look into the influence of quantum group in constructing the R-matrix solution of the QYBE (2.2). We may choose the ansatz $PR = \sum w^i \sigma^i \otimes \sigma^i$, P being the permutation operator $P = \sum_{i=0}^{3} \sigma^i \otimes \sigma^i$ and in analogy the operator L_n in the form $L = \sum_{i=0}^{3} w^i S^i \otimes \sigma^i$. It is interesting to note that specific algebraic properties of the operators S^i through (2.2) induce different sets of equations for $a = (w^0 + w^3)$ and $b = (w^0 - w^3)$ with $w^1 = w^2 = \frac{1}{2}$. In particular if S^i's are the generators of the SU(2) group with arbitrary spin having the property $[S^0, S^\alpha] = 0$, $[S^3, S^\pm]_+ \neq 0$, QYBE leads to two sets of equations like

$$a''b - b''a = b' \text{ and } (a'' - b'')(a - b) = a' - b' \tag{a, b}$$

etc. where we have used the notation '"' for argument $\lambda - \mu$, '′' for μ and undashed for λ However for $s = \frac{1}{2}$ representation with $[S^3, S^\pm]_+ = 0$ equations like (b) vanish altogether leaving only the set (a).

On the other hand, if S^i's belong to the trigonometric Sklyanin algebra (TSA)

$$[S^0, S^3] = 0, \quad [S^0, S^\pm] = \mp \tan^2 \frac{\alpha}{2} [S^\pm, S^3]_+, \tag{2.5}$$

$$[S^3, S^\pm] = \pm[S^0, S^\pm]_+, \; [S^+, S^-] = 4S^0S^3$$

related [5] to the quantum group $SU_q(2)$ with $q = e^{i\alpha}$, the resultant equations are given by same set (a), while (b) is replaced by another set

$$a''(a - \cos\alpha b) + b''(b - \cos\alpha a) = a' - \cos\alpha b'. \tag{c}$$

An easy check leads to the result that equations (a) and (b) allow the simple solution

$$a = \frac{\lambda + \mu}{\eta}, \; b = \frac{\lambda}{\eta} \tag{2.6a}$$

while (a) and (c) yield just a q-extension of it :

$$a = \left[\frac{\lambda + \eta}{\eta}\right]_q, \; b = \left[\frac{\lambda}{\eta}\right]_q \tag{2.6b}$$

with $[x]_q = \frac{\sin \alpha x}{\sin\alpha}$, whereas eq. (a) admits both (2.6a, b) as solutions. The above intriguing facts lead to the important conclusion that for SU (2) with arbitrary s representation one gets uniquely the solution (2.6a), which evidently results rational R_{xxx}-matrix, while for the associated quantum group SU_q (2) (naturally with arbitrary s) (2.6b) is the only solution giving the trigonometric R_{xxz}-matrix. Obviously at $q \to 1$, i.e., $\alpha \to 0$, when the quantum group is transformed to SU (2), the solution (2.6b) reduces to (2.6a) showing R_{xxz} to be a mere q-extension of R_{xxx}. Since at $s = 1/2$ limit set (a) is the only defining equation, it allows both the above rational and trigonometric solutions, which correspond to XXX and XXZ spin chains, respectively. This is also apparent from the fact that spin-1/2 representation of both SU (2) and SU_q (2) are identical. As the field models are associated with infinite dimensional Hilbert space they should be generated from the arbitrary spin representation, while the spin-1/2 case corresponds to XXX or XXZ spin chains. Since both field models and spin systems originate from the same group structure at different limits of spin values, the related R-matrices are given by the same universal solutions (rational or trigonometric) of QYBE and thus should naturally coincide.

At this point we should note certain symmetries of eq. (2.2), which allow important generalisation of the above solutions for R as well as L operators. In particular YBE admits deformations of the trigonometric R matrix through matrices $\alpha, \beta, \gamma, \delta$ in the form [2] $R(\lambda) \to \alpha\beta\gamma\delta\, R(\lambda)$. Note that among these transformations $\alpha_{ij}^{kl} = e^{\tau(j \;\; l)\lambda}$ is particularly important in braid groups as well as in conformal theories, since the resultant R matrix :

$$R' = \alpha R = e^{-i\alpha\lambda} R - e^{i\alpha\lambda} R_+$$

$$R = \begin{pmatrix} q^{-1} & & & \\ & 1 & q^{-1} - q & \\ & 0 & 1 & \\ & & & q^{-1} \end{pmatrix} \quad R_+ = \begin{pmatrix} q & & & \\ & 1 & 0 & \\ & q - q^{-1} & 1 & \\ & & & q \end{pmatrix}$$

possess the quantum group symmetry [5]. Similarly we would see below that transformation $\beta_{ij}^{kl} = e^{\theta(j \;\; k)}$ also has some relevance in integrable theories. On the other hand it may be observed that for the same R matrix (2.6b) QYBE (2.2) may yield a more general L operator solution given by the form

$$L = \begin{pmatrix} \frac{1}{\xi} I_1 + \xi I_2 & I_- \\ I_+ & \frac{1}{\xi} I_3 + \xi I_4 \end{pmatrix} \tag{2.7}$$

with $\xi = e^{i(\alpha\lambda/\eta)}$, where the generators satisfy now the extended Sklyanin algebra

$$[I_+, I_-] = -2i \sin\alpha\,(I_1 I_4 - I_2 I_3), \quad I_a I_\pm = e^{\pm i\varepsilon\alpha} I_\pm I_a, \tag{2.8}$$

where $\varepsilon = 1$ for $a = 1, 4$ and $\varepsilon = -1$ for $a = 2, 3$ with all I_i, $i \in [1, 4]$ commuting among themselves. It is not difficult to check that the above algebra goes to TSA at

i) $I_1 = -I_4$ and $I_2 = -I_3$. However, we find that it is the generators of algebra (2.8) that play a key role in formulating an unifying scheme for integrable models, because of its more general structure. Apart from TSA it allows also the reductions like

ii) $I_2 = I_3 = 0$, $I_1 = I_4$ with quadratic algebra $[I_+, I_-] = -2i \sin\alpha I_1^2$,

iii) $I_1 = -c\, I_2^{-1} = I_3^{-1} = c^{-1} I_4$ with algebra $[I_+, I_-] = -2ic \sin\alpha\,(I_1^2 + I_1^{-2})$,

iv) $I_2 = I_4 = 0$, $I_1 = I_3^{-1}$, yielding $[I_+, I_-] = 0$,

while $I_1 I_\pm = e^{\pm i\alpha} I_\pm I_1$ holds for all the above cases. Our investigation [6] leads to the interesting conclusion that the generators of the above reduced algebras related to the corresponding reductions of solution (2.7) through proper bosonisations may generate the Lax operators of a number of quantum integrable lattice models. Apart from their own right as bonafide integrable systems they may represent exact lattice versions of the corresponding integrable field models, obtainable easily at the continuous limit. All the models obtained finally are integrable systems and satisfy the QYBE as consistent reductions of the universal exact solution with the same quantum R matrix. This explains therefore why a wide class of lattice as well as field models share the same R-marix (rational or trigonometric). Leaving out the detail [6] we just mention that reduction i) yields lattice SG as well as the famous sine-Gordon model : $u_{\mu\nu} = \frac{m^2}{\alpha} \sin \alpha u$. Similarly ii) generates lattice and also Liouville field model $u_{\mu\upsilon} = \frac{m^2}{\alpha} e^{\alpha\upsilon}$. This reduction also opens up the promising possibility of constructing integrable lattice models through recently discovered q-oscillators. On the other hand through reduction iii) one may obtain bosonic Thiring model and also a novel quantum solvable derivative NLS represented by the equation $i\psi_t = \psi_{xx} - i(\psi^{\dagger}\psi)\psi_x$. Likewise iv) leads to light-cone SG and light-cone Liouville models. This list may even be enlarged considerably if one takes into consideration the deformation of algebra (2.8) by β transformation defined above. Among such models the Ladik-Ablowitz model expressed through q-bosons is worth mentioning. Another class of integrable models may be generated similarly at the limit $\alpha \to 0$, when the underlying quantum algebra (2.8) is reduced to Lie-like algebra. The corresponding trigonometric R matrix transforms at this limit to the rational R matrix solution mentioned above. At different reductions of the resultant Lie algebra one may again generate a series of models. The most important of them being NLS model, the well known Toda chain, while β-deformation allows again to induct other classes of integrable models under this unifying scheme. The generalisation of quantum algebra (2.8) for higher groups should also be able to generate models like Toda field theories, Vector NLS, N-wave solutions etc.

Therefore it is emphatically apparent that there exists an underlying symmetry related to the Sklyanin-like algebras, which in turn are determined by the universal solutions of QYBE. This symmetry represent an unifying thread in the labyrinth of diverse integrable world.

3. Conformal theory and the underlying quantum group symmetry

After the phenomenal work of BPZ [7] there has been a tremendous upsurge in the investigation of conformal field theory and related subjects. These theories exhibit a beautiful symmetry expressed through invariance under coordinate transformation $\xi^a \to \eta'(\xi)$ resulting tracelessness of the stress-energy tensor $T_a{}^a(\xi) = 0$. Another consequence is the conformal transformation of the metric tensor $g_{ab} \to \frac{\delta\varepsilon^{a'}}{\partial\eta^a}\frac{\delta\xi^{b'}}{\partial\eta^b} g_{a'b'} = \rho(\xi)\, g_{ab}$. In two dimensions such transformations constituting the conformal group are endowed with additional beauty. In this case conformal group becomes infinite dimensional and consists of analytic transformations : $z \to f(z)$, $\bar{z} \to \bar{f}(\bar{z})$ for the complex coordinates z and $\bar{z}$. One of the advantegious features of CFT is that the set $\{\phi_n(\xi)\}$, which includes identity I, the local field $\phi_n(\xi)$ and all its coordinate derivatives represent a closed operator algebra

$$\{\phi_n(\xi)\}\,\{\phi_m(0)\} = \sum C^k{}_{nm}(\xi)\,\{\phi_k(0)\}. \tag{3.1}$$

Hence the main problem boils down to finding only the c-number functions $C^k{}_{nm}$, which can be reduced further and expressed only by numerical parameters, expressible through anomolous dimensions Δ_i of the fields involved.

We would present here in brief some important features of CFT and try to focus on the deep rooted symmetry of such theories given through quantum groups. One of the most significant features of the 2-d CFT is that there are only two independent components T, $\bar{T}$ of the stress-energy tensor induced by its tracelessness condition, which turn out to be analytic functions $T = T(z)$, $\bar{T} = \bar{T}(\bar{z})$, i.e. become functions of single variable only. As a consequence one gets

$$\partial_{\bar{z}} < T(z) X > \; = \; 0, \qquad \partial_z < \bar{T}(\bar{z}) X > \; = \; 0, \tag{3.2}$$

i.e. the correlation functions $< T(z) X >$, with $X = \phi_{n_i}(\xi_i) \ldots \ldots \phi_{n_k}(\xi_k)$ is analytic function of z and regular everywhere except pole type singularities at $z = \xi_k$. Moreover, since under the infinitesimal conformal transformation we have (for $z \to z + \varepsilon(z)$)

$$\begin{aligned} \partial_\varepsilon \phi_j(z, \bar{z}) &= \oint_{c_z} d\xi \, \varepsilon(\xi) \, T(\xi) \, \phi_j(z, \bar{z}) \\ &= \varepsilon(z) \frac{\partial}{\partial z} \, \phi_n(z, z) + \Delta_n \varepsilon'(z) \, \phi_n(z, \bar{z}), \end{aligned} \tag{3.3}$$

with contour C_z surrounding the point z, the fields $T(z)$ (and similarly $\bar{T}(\bar{z})$) repesent the generators of the conformal group with the transformation

$$\delta_\varepsilon T(z) = \oint d_\xi \, \varepsilon(\xi) \, [T(\xi), T(z)] = \varepsilon(z) T'(z) + 2\varepsilon'(z) T(z) + \frac{c}{12} \varepsilon'''(z). \tag{3.4}$$

The above relation on the other hand through the introduction of operators L_n $(\bar{L}_n)$ as $T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}$ yield nothing other than the famous Virasoro algebra

$$[L_n, L_m] \; = \; (n-m) L_{n+m} + \frac{c}{12} (n^3 - n) \, \delta_{n+m,0} \tag{3.5}$$

with similar relations also for $\bar{L}_n$; L_n, $\bar{L}_n$ being commuting operators.

Note that L_α, $\alpha = 0, \pm 1$ constitute a closed subalgebra $sl(2, c)$ with the operator $H = L_0 + \bar{L}_0$ playing the role of Hamiltonian. In CFT there are some fundamental or seed fields ϕ_n called *primary fields* with dimensions Δ_n. Each of them under the application of L_{-k} may generate an infinite series of *secondary fields* given in the form

$$\phi^{(k_1, \ldots k_N)}(z) \; = \; L_{-k_1}(z) \ldots \ldots L_{-k_N} \phi_n(z) \tag{3.6}$$

The corresponding conformal dimensions of the secondary fields are given by $\Delta_n^{(k)} = \Delta_n + \sum_{i=1}^{N} k_i$ which may be easily checked using the property $L_0 \mid n > \; = \Delta_n \mid n >$, where $\mid n > \; = \phi_n(0) \mid 0 >$ and the obvious consequence of the Virasoro algebra (3.5): $L_0 L_{-k} \mid n > \; = L_{-k} L_0 \mid n > + \, k L_0 \mid n >$. Keeping in mind the property of the vaccum $L_m \mid n > \; = 0$ for $m > 0$ one concludes that the conformal family $\{\phi_n\}$ is isomorphic to the space of states generated by the primary and all its secondary fields. This space known as *Verma modulus* V_n gives a representation of the Virasoro algebra, which is irreducible in general.

Analogous also is the case with $\phi_n(\bar{z})$ having dimension $\bar{\Delta}_n$. Therefore the representation of $\{\phi_n\} = V_n \otimes \bar{V}_n$ is given by the string of states in the form $\mid n^{\{k\}\{\bar{k}\}} > \; = \prod_{n} L_{-k_i} \sum_{j} \bar{L}_{-k_j} \mid n >$. The Ward identity given by

$$<T(z)\,\phi_1(z_1)\ldots\phi_N(z_N)> = \sum_{j=1}^{N}\left\{\frac{\Delta_j}{(z-z_j)^2}+\frac{1}{(z-z_j)}\frac{\partial}{\partial z_j}\right\}<\phi_1(z_1)\ldots\phi_N(z_N)> \quad (3.7)$$

in such theory arises from the transformation property (3.3) and gives us the handle to reduce the correlators $<T(\xi_1)\ldots T(\xi_M)\,\phi_1(z_1)\ldots\phi_N(z_N)>_r$ through only $<\phi_1\ldots\phi_N>$. Similar situation occurs also in case of correlation function for secondary fields. Thus the correlators of primary fields contain the necessary information for all other correlators. Another interesting fact about this theory is that the operator product of primary fields $\Phi_n(z_1,\bar{z}_1)\,\Phi_m(z_2,\bar{z}_2)$ are expressed through bilocal operators $\psi_p(z_1,\bar{z}_1\,|\,z_2,\bar{z}_2)$ which in turn may be constructed from secondary fields generated through primary ϕ_p. the range of p is determined by the dimensions Δ_n and Δ_m. In particular for $n=m$:

$$\phi_n(z,\bar{z})\,|\,n> \;=\; \sum C^p_{nm}\, z^{\Delta_p\;2\Delta_n}\,\bar{z}^{\bar{\Delta}_p-2\bar{\Delta}_n}\,\psi_p(z\,\bar{z}\,|\,00), \quad (3.8)$$

where the field ψ_p is expressed through the states of the Verma modulus as

$$\psi_p(z,\;\bar{z}\,|\,0,0) \;=\; \sum z^{\Sigma K_i}\,\bar{z}^{\Sigma K_i}\,\beta_{nn}^{\Delta_p\{K\}}.\,\bar{\beta}_{nn}^{\Delta_p\{k\}}\,(L_{k_1}\ldots L_{-k_N}\,\bar{L}_{\bar{k}_1}\ldots\bar{L}_{\bar{k}_N})\,|\,p>$$

Using this relation, more complicated correlation functions may be computed. For example, the four point correlator $<\phi_k(\xi_1)\,\phi_e(\xi_2)\,\phi_n(\xi_3)\,\phi_{\epsilon m}(\xi_4)>$ may be expressed finally through conformal blocks $\widetilde{F}_{nm}^{\;lk}(p\,|\,x)$ given in trrms of

$$<k\,|\,\phi_e(1,1)\;\prod_{i=1}^{N}L_{-k_i}\,|\,p>.$$

In general the representation of the Virasoro algebra expressed through Verma modulus V_Δ is irreducible. However, an interesting situation occurs when the dimensions take some special values parametrised by two positive integers n and m as

$$\Delta_{(n,\,m)} \;=\; \Delta_0+\frac{1}{4}(\alpha_+ n+\alpha\, m)^2, \quad (3.9)$$

where $\Delta_0=\frac{c-1}{24}$ and $\alpha_\pm=\frac{1}{\sqrt{24}}\,[(1-c)^{1/2}\pm(25-c)^{1/2}]$. In this case there exists a secondary field $\chi(z)$ of dimension $\Delta_{(n,\,m)}+nm$, which behaves like primary field in the sense of commutation relations mentioned above but with zero norm $<\chi\,|\,\chi>=0$. Such null vectors make $V\Delta_{(n,\,m)}$ reducible which however, can be made irreducible again by demanding the family $\{\chi_{\Delta+nm}\}=0$. The resultant set $\{\phi_\Delta\}$ of the original primary fields, which contains now less number of fields are called *degenerate* conformal family.

The operator product expansions of two degenerate fields exhibit extremely intriguing properties. As for example, when two such fields are 'fused' together the result is

$$\Phi_{(n_1,\,m_1)}\,\Phi_{(n_2,\,m_2)} \;=\; \sum_{k=|n_1-n_2|+1}^{n_1+n_2-1}\;\;\sum_{e=|m_1-m_2|+1}^{m_1+m_2-1}\;\{\Phi_{(k,\,e)}\},$$

showing that degenerate conformal families form a closed operator algebra. Therefore CFT, where all primary fields are degenerate become interesting theories. Moreover when

$\frac{\alpha}{\alpha_{\prime}} = \frac{p}{q} = \tan\theta$, p, q being positive integers, the central charge becomes rational number $c = 1 - 6(p-q)^2/pq < 1$ and the conformal families $\{\phi_{(n,m)}\}$, $0 < n < p$, $o < m < q$ with finite number of primary fields form a closed algebra. Such theories which possess along with the usual truncation from below also this truncation from above, are known as *minimal or rational* CFT (RCFT). Note that $\psi_{(1,2)}$ and also $\psi_{(2,1)}$ act as shift operators $\varphi_{(1,2)}\phi_{(n,m)} = \{\phi_{(n,m-1)}\} + \{\phi_{(n,m+1)}\}$. Let us now look into the case $\frac{p}{q} = \frac{3}{4}$ corresponding to $c = \frac{1}{2}$, which allows only three kinds of primary fields in the theory, e.g. $\phi_{(1,1)} = \phi_{(2,3)} = I$ with $\Delta = 0$, $\phi_{(2,1)} = \phi_{(1,3)} = \varepsilon$ with $\Delta = \frac{1}{2}$ and $\phi_{(1,2)} = \phi_{(2,2)} = \sigma$ with $\Delta = \frac{1}{16}$. It is note-worthy that this minimal theory describes the critical limit of 2-*d* Ising model with fields σ, ε and I representing the local spin, energy density and identity operators, respectively.

Existence of such reduction of originally irreducible representation of Virasoro algebra in RCFT as well as the properties like truncation from above, 'fusion rule' etc. undoubtadely point towards the striking similarity with the representation of quantum groups with $q^n = 1$ and their comultiplication properties [3]. This also clearly signals an underlying quantum group structure in such theories at par with that of integrable models discussed above. In the case of quantum group with $q^n = 1$ this symmetry becomes more prominent, when nilpotent operators $(S^+)^n = 0$ with $< n \mid n > = 0$ appear resembling the null vectors of RCFT. As a consequence all the states with spin $j > \frac{(n-1)}{2}$ vanish, since the dimension of the representation is $[2j+1]_q$, while $[n]_q = 0$. This fact of irreducible representations turning reducible and breaking up into closed blocks (known as type II representation [5]) in quantum group is interestingly analogous to the picture of primary fields in RCFT discussed above. Such resemblance is perhaps best demonstrated in the $SU(2)$ Kac-Moody algebra related WZW model at level $k = n - 2$ [8]. It is known that the Clebsch-Gordan decomposition for the product of the quantum groups are given by

$$[j_1] \otimes [j_2] = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2,k-j_1-j_2)} [j]$$

which reduces to the standard Lie Algebra at $k \to \infty$. On the other hand the WZW model given by the primary fields $|j> = \Phi_j(0) \mid 0>$ exhibit exactly the same type of fusion rule. The representation of quantum group ρ_j (similar to $\phi_{(1,2)}$ in RCFT) acts for $j = \frac{1}{2}$ also as the shift operator $\rho_j \otimes \rho_{1/2} = \rho_{j-1/2} \otimes \rho_{j+1/2}$. Such similarities arise again in the fusion rule of conformal blocks $F_{\{a\}} = <0 \mid \phi_{a_1}(z_1)\ldots\ldots\phi_{a_n}(z_n) \mid 0>$. The R-matrix related to such blocks are found to be the elements of the braid groups [8], which are again the limiting cases ($\lambda \to \infty$) of the Yang-Baxter equation generating the quantum group.

4. Interrelation between integrable and conformal theories

Through above discussions we hope to demonstrate to some extent the intriguing connection of quantum groups with both the integrable and conformal theories. This common symmetry indicates also a deep interrelationship between these two aspects of 2-d theories. Though this connection has not been fully unearthed, the beautiful idea of perturbed CFT giving integrable systems (IS) and the fascinating C-theorem introduced by Zamolodchikov [9] are some positive steps towards it.

C-theorem tells us that in general in 2-d theories with $\Theta \neq T_{z\bar{z}} \pm 0$, one may define some function of the coupling constant as $C(g) = 2F - G - \frac{3}{4}H$, where F, G and H are related to the correlators of $T = T_{zz}$ and $\theta = T_{z\bar{z}}$ as

$$<T, \tilde{T}> = \frac{F(R)}{z^4}, \quad <T, \tilde{\theta}> = \frac{G(R)}{z^3\bar{z}}, \quad <\theta, \tilde{\theta}> = \frac{H(R)}{z^3\bar{z}}$$

with $T = T(z, \bar{z})$ and $\tilde{T} = T(0, 0)$, $R = (z, \bar{z})^{1/2}$. Now using conservation laws we may show that $C(g)$ satisfies the equation

$$R\frac{\partial C}{\partial R} = \frac{\partial C}{\partial e} = -\frac{3}{4}H < 0,$$

since $H > 0$. This simple relation has an interesting implication stating that in the space of coupling constants (theory space) the value of C along the renormalisation group trajectory is always decreasing. At fixed points corresponding to CFT's when $\theta = 0$ resulting $G = H = 0$, one gets $C(0) = c$ coinciding with the central charge of the theory. This beautiful picture however, asks for the explanation seeking what would happen if the CFT is perturbed and one tries to go away from criticality. Zamolodchikov's another conjecture [9] tries to answer this by stating that CFT through relevant perturbations might yield hierarchies of integrable systems. For example, $c = \frac{1}{2}$ CFT perturbed by the field $\sigma = \Phi_{(1}$, in the way $H = H_{1/2} + h\int \sigma(x)\, d^2x$, h being the dimensional constant, represents in fact the Ising model at $T = T_c$ with nonvanishing magnetic field h. Similarly perturbations of the WZW model (with level k) or the free scalar field model with $c = 1 - \frac{6}{p(p+1)}$, perturbed by the operator $\phi_{(1,3)}$ results generation of integrable SG model. However, as it has been mentioned above SG is related to quantum group with $q = e^{i\alpha}$, while WZW model to that with $q = e^{(2i\pi)/(K+2)}$. Therefore, one should expect to get some special SG as a result of perturbed WZW model. Actually this is the result one obtains and the SG thus generated is known as the restricted SG (RSG) [10].

This picture though interesting, is one-sided in some sense, i.e. depicts transition only from CFT to IS. We would therefore, like to look into such picture from the opposite direction, where the underplay of quantum group is more explicit. For understanding the metamorphosis of such transitions it is crucial to identify certain parameters in integrable systems, which are typical to IS and controls such possible transitions. In statistical models criticality is usually stated to be reached through temperature T at T_c. What is the corresponding parameters in integrable field theories ?

In (1 + 1) dimensions integrable models are associated with some linear systems given through Lax operators U_i, $i = 0, 1$ as $\partial_i \psi = U_i \psi$, $\psi \equiv (\psi_1, \psi_2)$ with the flatness condition

$$\partial_i U_j - \partial_j U_i + [U_i, U_j] = 0,$$

which yields the given integrable nonlinear equation. Jost solution ψ_1 with the asymptotic plane wave solution for $\psi_1(-\infty)$ is connected with the spectral data $a(\lambda)$ as $\psi_1^{(x)} \xrightarrow[x \to +\infty]{} a(\lambda)\, \psi_1(-\infty)$. In general for any path P_{12} in the plane connecting the points p_1 and p_2 we may write

$$\psi_1(p_2) = P_{12} \exp \left(\int A_z\, dz + A_{\bar z}\, d\bar z\right) \psi_1(p_1) \tag{4.1}$$

where $\vec{A} = (A_z, A_{\bar z})$ with $z = x^0 + x^1$; $\bar z = x^0 - x^1$, a vector function defined in the extended space with the inclusion of spectral parameter λ. The integrability condition yields

$$\partial_{\bar z} A_z (z, \bar z, \lambda) + \partial_z A_{\bar z} (z, \bar z, \lambda) = 0, \tag{4.2}$$

assuring the path-independence of the integral. Stretching now points p_1 and p_2 to $-$ and $+\infty$, respectively, one easily obtains

$$\ln a(\Lambda) = \sum_{-\infty}^{\infty} (A_z dz + A_{\bar z} d\bar z), \tag{4.3}$$

If now we expand these objects as (2.4a, b) denoting $A_z = \sum_n T^{(n)}/\lambda^n$, $\lambda \to \infty$ and $A_z = \sum_n \theta^{(n)} \lambda^n$, $\lambda \to 0$ (and similarly for $A_{\bar z}$), then from (4.3) one immediately gets conserved quantities as

$$C_n = \int_{-\infty}^{\infty} (T^{(n)}\, d_z + \bar\theta^{(n)} d_z), \quad D_n = \int_{-\infty}^{\infty} (\bar T^{(n)}\, d_{\bar z} + \theta^{(n)} d_z) \tag{4.4}$$

with the flatness condition (4.2) reducing to

$$\partial_{\bar z} T^{(n)} + \partial_z \bar\theta^{(n)} = 0, \qquad \partial_{\bar z} \theta^{(n)} + \partial_z \bar T^{(n)} = 0 \tag{4.5}$$

We may observe also that the Hamiltonian (H) and momentum (P) in relativistic models (with canonical dimension 1 in 2-d theories) should be constructed from C_1 and D_1. However on the other hand, since $\ln a(\lambda)$ is dimensionless and so is λ for relativistic models, C_n and D_n, as a consequence, must be dimensionless ! The remedy from this apparent paradox is that a dimensional mass parameter m must always be present, which must enter the conserved charges making them dimensional.

Now looking into the structure of A_i for integrable relativistic models from the Lorentz invariance in light-cone coordinates and dimensional consideration, we may conclude that in the simplest case (e.g. for SG, Liouville, WZW models) the mass parameter m and the spectral parameter λ enter into their structure in the combination $\frac{m}{\lambda}$ and $m\lambda$ respectively. Let us now look into the limit $m \to 0$, when the system approaches its conformal limit and examine closely the transition process. Firstly, it is immediate that for some nontrivial evolution of the system one may set either of the two limits $\lambda \to \infty$ or $\lambda \to 0$. In the first case $m\lambda < \infty$, which keeps A_z nonvanishing resulting nontrivial evolution along z, one unavoidably gets $\frac{m}{\lambda} \to 0$ leading to vanishing of $A_{\bar{z}}$, i.e. the vanishing $\bar{z}$ evolution. Similar situation repeats in the second case when $\lambda \to 0$, interchanging the role of z and $\bar{z}$. These limits of the spectral parameter induce interesting consequences. As for example, at $\lambda \to \infty$ when the evolution along $\bar{z}$ direction is frozen : $\partial_{\bar{z}}\psi = 0$, any arbitrary points (p_1, p_2) can not be joined now contrary to the integrable case discussed above. Expression (4.1) consequently reduces to the form $\psi_1 \exp \int A_z dz$, the allowed path being now only along the z axis, the slope of it naturally being fixed. It is like particles moving with a fixed velocity equal to the velocity of light, which is also required by the masslessness condition. Exactly similar situation repeats again for the complementary case $\lambda \to 0$, when the roles of z and $\bar{z}$ are reversed with $A_z = 0$ and nontrivial $A_{\bar{z}}$. Amazingly in both these cases i.e. for $\lambda \to \infty$ and $\lambda \to 0$, even the classical soliton solution with velocity $V = c\,\frac{1 - |\lambda_1|^2}{1 + |\lambda_1|^2}$ starts moving with the velocity of light c. This transition process to the conformal limit also affects the analytic structure of $a(\lambda)$ in an interesting way. Since at $\lambda \to \infty$ the analyticity of $a(\lambda)$ remains only at this limit with its expansion (2.4) now valid only at this point, resulting nontrivial set of C_n, while vanishing of all the conserved quantities D_n, the truncation of this whole series reduces the conservation laws (4.5) typical to the IS to $\partial_{\bar{z}} I^{(n)} = 0$ typical for CFT. On the other hand at these particular limits only one gets the equality $\lambda = \mu = \lambda + \mu$, which, as is well known, reduces YBE

$$R_{12}(\lambda - \mu)\, R_{13}(\lambda)\, R_{23}(\mu) = R_{23}(\mu)\, R_{13}(\lambda)\, R_{12}(\lambda - \mu),$$

the characteristic equation of IS into the braid group relation

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12},$$

representing the quantum group structure in CFT. At these limits the trigonometric quantum R-matrix of integrable models also turns into the solutions of braid group $R_{\pm}$ respectively.

5. Concluding remarks

The main characteristics of Integrable systems and conformal field theory are reviewed giving special emphasis on their underlying quantum group structures. The interrelationship between these two branches of 2-d theories has been highlighted with an attempt to analyse the metamorphosis of transition from IS to CFT. At vanishing mass parameter corresponding to some limits of the spectral parameter λ, the CFT models result from IS. Zamolodchikov's conjecture affirms on the other hand that IS is obtainable from CFT through relevant perturbations. This close interrelation possibly is the reason why both these theories share the same inbuilt quantum group structure. The difference however being that, in IS the quantum group is usally more prominent, since the models may be defined on lattices, where exact q-group related Sklyanin algebra is represented exactly. On the other hand in CFT, which is obtained in the continuous and infinite interval limit we are able to see only the remnants of the quantum group, occasionally discovered by introducing assymetry through boundary conditions [8].

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